

A Consistent BGK-Type Model for Gas Mixtures

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Received July 16, 2001; accepted September 24, 2001

We introduce a relaxation collision operator for a mixture of gases which satisfies several fundamental properties. Different BGK type collision operators for gas mixtures have been introduced earlier but none of them could satisfy all the basic physical properties: positivity, correct exchange coefficients, entropy inequality, indifferenciability principle. We show that all those properties are verified for our model, and we derive its Navier–Stokes limit by a Chapman–Enskog expansion.

KEY WORDS: Kinetic theory; gas mixtures; BGK-type models; entropy; Chapman–Enskog expansion.

1. INTRODUCTION

It is well known that the relevant mathematical equation for the evolution of rarefied or high altitude gases is the Boltzmann equation.

Since this equation is rather complex, various simpler models have been introduced and are widely used. The most famous one is the BGK model^(1,2) which replaces the collision operator with the relaxation towards a Maxwellian and verifies some basic properties of the Boltzmann equation. Moreover an amelioration of the BGK model, the so called ES-BGK model (see ref. 3 for a recent complete study) gives the correct thermoconductivity and viscosity coefficients in the fluid limit.

Most of these models have nevertheless been considered in the case of a single species gas. This seems an important limitation recalling for example

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that the atmosphere must at least be considered as a binary mixture of Oxygen and Nitrogen. As recent applications where gas mixtures are considered on the basis of kinetic theory, let us mention the evaporation-condensation problems where species behave differently on the boundary⁽⁴⁾ or the mixtures in nuclear engineering.⁽⁵⁾

Although the extension of the Boltzmann equation to a mixture of gases has been well known for a long time, this is not the case for the BGK equation. In fact two of the authors of the classical one species BGK model have also introduced a similar model for mixtures in ref. 6. Other models were introduced thereafter, see, for example, ref. 7, whereas a generalization of ref. 6 to various interaction potentials is obtained in ref. 8. But as pointed out by ref. 9 all these models have an important drawback: when all the species are identical one does not recover the equation for a single component gas (this *indifferentiability principle* is verified by the Boltzmann equation). In ref. 9 a model which verifies this principle is introduced but the positivity is lost.

In this paper we propose a model which overcomes these difficulties, while recovering the correct exchange coefficients. The main idea is that instead of approximating each of the binary collision operators (between species i and j) by a BGK-type equation (or some kind of moment closure if the species are different in ref. 7), we introduce only one “global” (i.e., taking into account all the species j) operator for each species i .

The outline of this paper is the following: we first recall the main properties of the Boltzmann collision operator, we then introduce our BGK-type model and prove that it satisfies the expected properties (positivity, exchange coefficients, entropy inequality, indifferentiability principle). We give also the Euler and Navier–Stokes systems associated with our model.

2. NOTATIONS AND PROPERTIES OF THE BOLTZMANN OPERATOR

2.1. The Boltzmann Equation

The Boltzmann collision operator for mixtures is written for $t \geq 0$ and $x \in \mathbb{R}^3$,

$$Q_i(f, f) = \sum_{k=1}^n Q_{ik}(f_i, f_k) \tag{2.1}$$

$$Q_{ik}(f_i, f_k) = \int_{\mathbb{R}^3} \int_{B_+} (f'_i f'_{k^*} - f_i f_{k^*}) B_{ik}(n, V, |V|) d\xi_* dn$$

Here ξ is the molecular velocity, ξ_* is an integration variable, n is a unit vector and B_+ is the semi sphere defined by $n \cdot V = 0$, where V is the relative velocity

$$V = \xi - \xi_*$$

The post collisional velocities are

$$\begin{cases} \xi' = \xi - \frac{2\mu_{ik}}{m_i} n [(\xi - \xi_*) \cdot n] \\ \xi'_* = \xi_* + \frac{2\mu_{ik}}{m_k} n [(\xi - \xi_*) \cdot n] \end{cases} \quad (2.2)$$

and the reduced mass is

$$\mu_{ik} = m_i m_k / (m_i + m_k) \quad (2.3)$$

We recall that these rules are dictated by the conservation laws for momentum and energy

$$\begin{aligned} m_i \xi + m_k \xi_* &= m_i \xi' + m_k \xi'_* \\ m_i |\xi|^2 + m_k |\xi_*|^2 &= m_i |\xi'|^2 + m_k |\xi'_*|^2 \end{aligned} \quad (2.4)$$

We also recall that the micro-collision operator

$$T_n: (\xi, \xi_*) \mapsto (\xi', \xi'_*)$$

is an involution

$$T_n \circ T_n = I_{6 \times 6}$$

in other words $(\xi')' = \xi$, $(\xi'_*)' = \xi_*$. This property and direct computations show the following identities, which are fundamental for the study of the properties of the collision operator,

$$\begin{aligned} d\xi \, d\xi_* &= d\xi' \, d\xi'_* \\ n \cdot (\xi - \xi_*) &= -n \cdot (\xi' - \xi'_*) \\ |\xi - \xi_*| &= |\xi' - \xi'_*| \end{aligned} \quad (2.5)$$

2.2. Macroscopic Quantities

We introduce notations for macroscopic quantities that will be used later on, $n^{(i)}$ is the number density, $\rho^{(i)}$ the density, $v^{(i)}$ the average velocity, $E^{(i)}$ the energy per unit volume, $\varepsilon^{(i)}$ the internal energy per particle, $T^{(i)}$ the temperature of species i .

$$n^{(i)} = \int_{\mathbb{R}^3} f_i d\xi, \quad \rho^{(i)} = m_i n^{(i)}$$

$$n^{(i)} v^{(i)} = \int_{\mathbb{R}^3} \xi f_i d\xi$$

$$E^{(i)} = \frac{1}{2} \rho^{(i)} |v^{(i)}|^2 + n^{(i)} \varepsilon^{(i)}, \quad \varepsilon^{(i)} = \frac{3}{2} k_B T^{(i)}$$

$$\varepsilon^{(i)} = \frac{m_i}{2n^{(i)}} \int_{\mathbb{R}^3} |\xi - v^{(i)}|^2 f_i d\xi$$

Then, we also have global quantities for the mixture, the total density ρ , the number density n , the mean velocity v , the energy per unit volume E and $n\varepsilon$ the internal energy per unit volume.

$$\rho = \sum_{k=1}^n \rho^{(k)}, \quad n = \sum_{k=1}^n n^{(k)} \quad (2.6)$$

$$\rho v = \sum_{k=1}^n \rho^{(k)} v^{(k)} \quad (2.7)$$

$$n\varepsilon + \frac{\rho}{2} |v|^2 = E = \sum_{k=1}^n E^{(k)} \quad (2.8)$$

2.3. Indifferentiability Principle

We call *indifferentiability principle* the following property:

When all the masses m_i and cross-sections B_{ik} are identical, the total distribution $f = \sum f_i$ obeys the single species Boltzmann equation.

This property is satisfied by bilinearity of the collision operator.

Notice also that the macroscopic quantities associated to f are n , v , ε .

2.4. Conservation Laws and Transfer Coefficients

For multispecies fluids, we have to keep in mind that for each species the usual mass, momentum and energy are not necessarily conserved. In

the present model, we suppose that there is no chemical reaction between the species, so that the mass of each species is conserved,

$$\int_{\mathbb{R}^3} Q_i d\zeta = 0$$

Each species may nevertheless exchange momentum and energy with the others. The exchange relations can be computed exactly in the case of maxwellian particles, and the computations are left in Appendix 1. See also ref. 8 for some extensions to other kind of particles.

One obtains

$$\int_{\mathbb{R}^3} m_i \zeta Q_i d\zeta = \sum_{k=1}^n 2\mu_{ik} \chi_{ik} n^{(i)} n^{(k)} [v^{(k)} - v^{(i)}]$$

$$\int_{\mathbb{R}^3} \frac{m_i}{2} |\zeta - v^{(i)}|^2 Q_i d\zeta = \sum_{k=1}^n 2\mu_{ik} \chi_{ik} n^{(i)} n^{(k)} \frac{2}{m_i + m_k} \left(\varepsilon^{(k)} - \varepsilon^{(i)} + m_k \frac{|v^{(k)} - v^{(i)}|^2}{2} \right)$$

where μ_{ik} is the reduced mass given in (2.3), and the interaction coefficient χ_{ik} is defined by

$$\chi_{ik} = \int_{B_+} (\cos \omega)^2 \bar{B}_{ik}(\omega) d\omega \quad (2.9)$$

where \bar{B}_{ik} is related to the interaction potential between species i and k .

Remark 2.1. The interaction coefficient χ_{ik} is smaller than the collision frequency ν_{ik} between species i and k , defined by

$$\nu_{ik} = \int_{B_+} \bar{B}_{ik}(\omega) d\omega \quad (2.10)$$

Especially for non cut-off models, ν_{ik} might be infinite while χ_{ik} remains finite.

2.5. Equilibrium

The equilibrium in the mixture is obtained when all $Q_i = 0$. Then a classical result is that every $Q_{ij} = 0$. In Appendix 2 we give a proof of the following:

If for two species i and j $Q_{ij} = Q_{ji} = 0$ then f_i and f_j are Maxwellians with common velocity and temperature.

The conclusion is that at global equilibrium, all distributions are

$$f_i = n^{(i)} \left(\frac{m_i}{2\pi k_B T} \right)^{3/2} \exp \left(-\frac{m_i |\xi - v|^2}{2k_B T} \right) \quad (2.11)$$

for some common velocity v and temperature T .

2.6. Entropy Inequality

In the case of a mixture, the entropy inequality (H theorem) is

$$\sum_i \int Q_i \ln f_i d\xi \leq 0$$

3. THE BGK-TYPE MODEL

Here, we propose a relaxation model which satisfies the properties

- the non-negativity of densities is satisfied,
- the exchange relations are those of the Maxwellian particle model,
- the “indifferentiability principle” holds,
- the equilibrium distributions are Maxwellians with common velocities and internal energies,
- the H theorem holds true.

3.1. Formulation of the BGK-Type Model

The model is built as follows. The relaxation occurs toward a maxwellian distribution M_i , i.e.,

$$\frac{\partial}{\partial t} f_i + \xi \cdot \nabla_x f_i = Q_i := \nu_i [M_i - f_i] \quad (3.1)$$

with the notation

$$M_i = n^{(i)} \left(\frac{m_i}{2\pi k_B T_i} \right)^{3/2} \exp \left(-\frac{m_i |\xi - v_i|^2}{2k_B T_i} \right) \quad (3.2)$$

where ν_i is a collision frequency and its choice is crucial (see Theorem 3.1 below).

The macroscopic parameters v_i and $\varepsilon_i = \frac{3}{2} k_B T_i$ are chosen to recover the exchange relations. This gives

$$m_i v_i v_i = m_i v_i v^{(i)} + \sum_{k=1}^n 2\mu_{ik} \chi_{ik} n^{(k)} [v^{(k)} - v^{(i)}] \tag{3.3}$$

$$v_i \varepsilon_i = v_i \varepsilon^{(i)} - \frac{m_i v_i}{2} |v_i - v^{(i)}|^2 + \sum_{k=1}^n 2\mu_{ik} \chi_{ik} n^{(k)} \frac{2}{m_i + m_k} \times \left(\varepsilon^{(k)} - \varepsilon^{(i)} + m_k \frac{|v^{(k)} - v^{(i)}|^2}{2} \right) \tag{3.4}$$

Theorem 3.1. The internal energy ε_i is positive as soon as

$$v_i \geq \sum_{k=1}^n \chi_{ik} n^{(k)} \tag{3.5}$$

In particular the model is well defined with the total collision frequency (see Remark 2.1)

$$v_i = \sum_{k=1}^n v_{ik} n^{(k)}$$

Proof of Theorem 3.1. The first condition for the positivity of ε_i is, taking the coefficient in front of $\varepsilon^{(i)}$ in (3.4),

$$v_i \geq \sum_{k=1}^n 2\mu_{ik} \chi_{ik} n^{(k)} \frac{2}{m_i + m_k} \tag{3.6}$$

Next we have to check

$$\sum_{k=1}^n 2\mu_{ik} \chi_{ik} n^{(k)} \frac{2m_k}{m_i + m_k} |v^{(k)} - v^{(i)}|^2 - v_i m_i |v_i - v^{(i)}|^2 \geq 0 \tag{3.7}$$

For this we use the formula for the exchange of momentum

$$v_i m_i (v_i - v^{(i)}) = \sum_{k=1}^n 2\mu_{ik} \chi_{ik} n^{(k)} [v^{(k)} - v^{(i)}] \tag{3.8}$$

and a Cauchy–Schwarz inequality

$$\left(\sum ab \right)^2 \leq \sum a^2 \sum b^2$$

Taking $a = \sqrt{2\mu_{ik}\chi_{ik}n^{(k)}\frac{m_i+m_k}{2m_k}}$ and $b = \sqrt{2\mu_{ik}\chi_{ik}n^{(k)}\frac{2m_k}{m_i+m_k}} [v^{(k)} - v^{(i)}]$ one obtains

$$|v_i m_i (v_i - v^{(i)})|^2 \leq \sum_{l=1}^n 2\mu_{il}\chi_{il}n^{(l)} \frac{m_i+m_l}{2m_l} \sum_{k=1}^n 2\mu_{ik}\chi_{ik}n^{(k)} \frac{2m_k}{m_i+m_k} |v^{(k)} - v^{(i)}|^2$$

The inequality (3.7) is satisfied if

$$\sum_{k=1}^n 2\mu_{ik}\chi_{ik}n^{(k)} \frac{2m_k}{m_i+m_k} |v^{(k)} - v^{(i)}|^2 \left(1 - \frac{1}{m_i v_i} \sum_{l=1}^n 2\mu_{il}\chi_{il}n^{(l)} \frac{m_i+m_l}{2m_l}\right) \geq 0$$

which gives

$$m_i v_i \geq \sum_{k=1}^n 2\mu_{ik}\chi_{ik}n^{(k)} \frac{m_k+m_i}{2m_k}$$

and the second condition

$$v_i \geq \sum_{k=1}^n \chi_{ik}n^{(k)} \tag{3.9}$$

To conclude, remark that

$$\frac{4\mu_{ik}}{m_i+m_k} = \frac{4m_i m_k}{(m_i+m_k)^2} \leq 1$$

so that the first condition is always verified when the second one is.

The last proposition of the theorem is only a consequence of the inequality $\chi_{ik} \leq v_{ik}$, see Remark 2.1.

3.2. Indifferentiability Principle

Proposition 3.2. Suppose that all masses are equal to m , all transfer coefficients are equal to χ and choose the collision rates v_i all equal to $v = \chi \sum_k n^{(k)} = \chi n$. Then the total distribution $f = \sum_k f_k$ verifies a BGK equation with collision rate v . Thus the indifferentiability principle holds.

Proof of Proposition 3.2. We first rewrite (3.3) in this case and obtain

$$m v v_i = m v v^{(i)} + \sum_{k=1}^n m \chi n^{(k)} [v^{(k)} - v^{(i)}]$$

Recalling the choice of v , one obtains

$$\rho v_i = \sum_{k=1}^n \rho^{(k)} v^{(k)}$$

so that each velocity v_i is equal to the “mean” velocity v (see Section 2.2).

Next, the same computations for the energy give

$$\begin{aligned} v \varepsilon_i &= v \varepsilon^{(i)} - \frac{mv}{2} |v - v^{(i)}|^2 + \sum_{k=1}^n m \chi n^{(k)} \frac{1}{m} \\ &\quad \times \left(\varepsilon^{(k)} - \varepsilon^{(i)} + m \frac{|v^{(k)} - v^{(i)}|^2}{2} \right) \\ n \varepsilon_i + \frac{mn}{2} |v - v^{(i)}|^2 &= \sum_{k=1}^n n^{(k)} \left(\varepsilon^{(k)} + m \frac{|v^{(k)} - v^{(i)}|^2}{2} \right) \end{aligned}$$

Since

$$\sum_{k=1}^n n^{(k)} v^{(k)} v^{(i)} = n v v^{(i)}$$

we obtain

$$n \left(\varepsilon_i + \frac{m}{2} |v|^2 \right) = \sum_{k=1}^n n^{(k)} \left(\varepsilon^{(k)} + \frac{m}{2} |v^{(k)}|^2 \right)$$

This means that the internal energies ε_i are identical to ε . To conclude, we can now write

$$\begin{aligned} M_i &= n^{(i)} \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp -\frac{m |\xi - v|^2}{2k_B T} \\ M &= \sum_i M_i = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp -\frac{m |\xi - v|^2}{2k_B T} \end{aligned}$$

Summing Eqs. (3.1) gives

$$\frac{\partial}{\partial t} f + \xi \cdot \nabla_x f = Q := v[M - f]$$

3.3. Equilibrium

At global equilibrium, it is evident that all distribution functions are Maxwellians, but it is not so evident that velocities and internal energies are all the same.

Proposition 3.3. The equilibrium distributions for the model (3.1) are Maxwellians with common velocity and internal energy.

Proof of Proposition 3.3. From the definition of v_i (3.3) we obtain

$$v_i = \frac{1}{m_i v_i} \sum_{k=1}^n 2\mu_{ik} \chi_{ik} n^{(k)} v^{(k)} + \left(1 - \frac{1}{m_i v_i} \sum_{k=1}^n 2\mu_{ik} \chi_{ik} n^{(k)} \right) v^{(i)}$$

This means that the velocity v_i is a strictly convex combination of the velocities of the species.

At equilibrium $v^{(i)} = v_i$ and each velocity is a strictly convex combination of all the others, it is easy to conclude that the $v^{(i)}$ (and thus v_i) are all equal.

Then rewriting in this case (3.4)

$$v_i \varepsilon_i = v_i \varepsilon^{(i)} + \sum_{k=1}^n 2\mu_{ik} \chi_{ik} n^{(k)} \frac{2}{m_i + m_k} (\varepsilon^{(k)} - \varepsilon^{(i)})$$

we obtain again that ε_i is a strictly convex combination of the internal energies, and the same argument applies.

3.4. Entropy Inequality

We begin with a lemma

Lemma 3.4. The following inequality holds

$$\sum_{i=1}^n n^{(i)} v_i \ln \varepsilon_i \geq \sum_{i=1}^n n^{(i)} v_i \ln \varepsilon^{(i)}$$

To prove this inequality, let us recall the formula (3.4) for ε_i

$$\begin{aligned} & v_i \left(\varepsilon_i + \frac{m_i}{2} |v_i - v^{(i)}|^2 - \varepsilon^{(i)} \right) \\ &= \sum_{k=1}^n 2\mu_{ik} \chi_{ik} n^{(k)} \frac{2}{m_i + m_k} \left((\varepsilon^{(k)} - \varepsilon^{(i)}) + m_k \frac{|v^{(k)} - v^{(i)}|^2}{2} \right) \end{aligned}$$

The contribution of the velocity terms is positive (with the condition on (3.5) on v_i) and one obtains

$$\varepsilon_i \geq \frac{1}{v_i} \sum_{k=1}^n 2\mu_{ik} \chi_{ik} n^{(k)} \frac{2}{m_i + m_k} (\varepsilon^{(k)} - \varepsilon^{(i)}) + \varepsilon^{(i)}$$

The right hand side is a convex combination of the energies. Because the logarithm is a concave increasing function, we can write

$$\ln \varepsilon_i \geq \frac{1}{v_i} \sum_{k=1}^n 2\mu_{ik}\chi_{ik}n^{(k)} \frac{2}{m_i + m_k} (\ln \varepsilon^{(k)} - \ln \varepsilon^{(i)}) + \ln \varepsilon^{(i)}$$

Then

$$\begin{aligned} \sum_{i=1}^n n^{(i)}v_i \ln \varepsilon_i &\geq \sum_{i=1}^n \sum_{k=1}^n 2\mu_{ik}\chi_{ik}n^{(i)}n^{(k)} \frac{2}{m_i + m_k} (\ln \varepsilon^{(k)} - \ln \varepsilon^{(i)}) \\ &\quad + \sum_{i=1}^n n^{(i)}v_i \ln \varepsilon^{(i)} \end{aligned}$$

It is straightforward to verify that the double sum vanishes (exchanging the indices and using the symmetry of χ) and the result follows.

Theorem 3.5. The following entropy inequalities hold true

$$\begin{aligned} \sum_{k=1}^n \int_{\mathbb{R}^3} v_k M_k \ln M_k \, d\xi &\leq \sum_{k=1}^n \int_{\mathbb{R}^3} v_k M^{(k)} \ln M^{(k)} \, d\xi \\ \sum_{k=1}^n \int_{\mathbb{R}^3} v_k M^{(k)} \ln M^{(k)} \, d\xi &\leq \sum_{k=1}^n \int_{\mathbb{R}^3} v_k f_k \ln f_k \, d\xi \end{aligned}$$

with $M^{(i)} = n^{(i)} \left(\frac{m_i}{2\pi k_B T^{(i)}}\right)^{3/2} \exp\left[-\frac{m_i |\xi - v^{(i)}|^2}{2k_B T^{(i)}}\right]$, and the equality holds only in the situation of a global equilibrium. Solutions of (3.1) satisfy the H theorem

$$\frac{\partial}{\partial t} \sum_{k=1}^n \int_{\mathbb{R}^3} f_k \ln f_k \, d\xi + \operatorname{div}_x \sum_{k=1}^n \int_{\mathbb{R}^3} \xi f_k \ln f_k \, d\xi \leq 0$$

Proof of Theorem 3.5. The first inequality can be written, after explicit calculations,

$$\begin{aligned} \sum_{k=1}^n v_k n^{(k)} \ln \left[n^{(k)} \left(\frac{m_k}{2\pi k_B T_k}\right)^{3/2} \right] - n^{(k)} \\ \leq \sum_{k=1}^n v_k n^{(k)} \ln \left[n^{(k)} \left(\frac{m_k}{2\pi k_B T^{(k)}}\right)^{3/2} \right] - n^{(k)} \end{aligned}$$

which is also equivalent to

$$\sum_{k=1}^n v_k n^{(k)} \ln T_k \geq \sum_{k=1}^n v_k n^{(k)} \ln T^{(k)}$$

a consequence of Lemma 3.4.

The second inequality is in fact true term by term:

$$\int_{\mathbb{R}^3} M^{(k)} \ln M^{(k)} d\xi \leq \int_{\mathbb{R}^3} f_k \ln f_k d\xi$$

is the H theorem for the standard monospecies BGK equation.

To conclude

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{k=1}^n \int_{\mathbb{R}^3} f_k \ln f_k d\xi + \operatorname{div}_x \sum_{k=1}^n \int_{\mathbb{R}^3} \xi f_k \ln f_k d\xi \\ = \sum_{k=1}^n \int_{\mathbb{R}^3} v_k (M_k - f_k) \ln f_k d\xi \end{aligned}$$

Denoting $H(x) = x \ln x - x$ and using the convexity relation

$$H'(f)(g-f) \leq H(g) - H(f)$$

it is sufficient to prove

$$\begin{aligned} \sum_{k=1}^n \int_{\mathbb{R}^3} v_k (H(M_k) - H(f_k)) d\xi \leq 0 \\ \sum_{k=1}^n \int_{\mathbb{R}^3} v_k (H(M_k) - H(M^{(k)}) + H(M^{(k)}) - H(f_k)) d\xi \leq 0 \end{aligned}$$

This is a consequence of the first inequalities.

4. CHAPMAN-ENSKOG EXPANSION

This section is devoted to the derivation of the Navier–Stokes system in the compressible regime (we refer to ref. 10 for other possible regimes in the one species case and ref. 11 for a complete discussion and references).

In order to derive the Navier–Stokes system, we assume there is a small parameter λ such that every $v_i = O(\frac{1}{\lambda})$ and we study the expansion of macroscopic quantities in term of λ .

Proposition 4.1. The solutions of (3.1) verify the system of $n+4$ equations

$$\partial_t \rho^{(i)} + \operatorname{div}(\rho^{(i)} v) = -\operatorname{div}(J_i) \tag{4.1}$$

$$\partial_t(\rho v) + \operatorname{div}(P + \rho v \otimes v) = 0 \tag{4.2}$$

$$\partial_t E + \operatorname{div}(E v + P \cdot v + q) = 0 \tag{4.3}$$

where ρ , v , and E are given by (2.6), (2.8), (2.8), and

$$J_i = m_i \int_{\mathbb{R}^3} f_i(\xi - v) d\xi = \rho^{(i)}(v^{(i)} - v) \tag{4.4}$$

$$P = \sum_i \int_{\mathbb{R}^3} m_i f_i(\xi - v) \otimes (\xi - v) d\xi \tag{4.5}$$

$$q = \sum_i \int_{\mathbb{R}^3} m_i f_i(\xi - v) \frac{|\xi - v|^2}{2} d\xi \tag{4.6}$$

The system is closed in the first order of λ as a system of the $n+4$ unknowns $\rho^{(i)}$, v , T by the following relations:

$$J_i = -\sum_{k=1}^n L_{ik} \frac{\nabla_x(n^{(k)} k_B T)}{\rho^{(k)}} + O(\lambda^2) \tag{4.7}$$

$$P = n k_B T \underline{\underline{Id}} - \eta(\nabla_x u + (\nabla_x u)^t) - \frac{2}{3}(\operatorname{div} u) \underline{\underline{Id}} + O(\lambda^2) \tag{4.8}$$

$$q = \frac{5}{2} k_B T \sum_i \frac{J_i}{m_i} - \kappa \nabla_x T + O(\lambda^2) \tag{4.9}$$

where L is a symmetric matrix whose coefficients depend only upon the masses and densities, η and κ are the viscosity and thermal conductivity coefficients

$$\eta = k_B T \sum_i \frac{n^{(i)}}{v_i} \tag{4.10}$$

$$\kappa = \frac{5}{2} k_B^2 T \sum_i \frac{n^{(i)}}{m_i v_i} \tag{4.11}$$

Remark 4.2. In classical manuals (see, for example, ref. 12), $\frac{\nabla_x(n^{(k)}k_B T)}{\rho^{(k)}}$ is denoted by the x derivative at constant temperature of the chemical potential:

$$\frac{k_B T}{m_k} \left(\ln n^{(k)} - \frac{3}{2} \ln \frac{2k_B T}{m_k} \right)$$

The symmetry of L is also called *Onsager relation*.

Remark 4.3. The system we obtain is exactly the same as the system obtained in ref. 9. In the case of the Boltzmann equation with Maxwellian particles, one obtains the same system, but with different coefficients η , κ and L_{ik} (see ref. 12).

Proof of Proposition 4.1. The system can always be written by integration of the kinetic equation (3.1) and is the conservation of the mass of each species, the total momentum and the total energy.

Thus we aim to prove the closure relations, this is done in the following subsections.

4.1. Euler Limit

At equilibrium the Euler system holds for the densities, common velocity and common temperature defined by (2.6), (2.8), (2.8),

$$\partial_t \rho^{(i)} + \operatorname{div}(\rho^{(i)} v) = 0 \quad (4.12)$$

$$\partial_t(\rho v) + \operatorname{div}(nk_B T \underline{Id} + \rho v \otimes v) = 0 \quad (4.13)$$

$$\partial_t E + \operatorname{div}((E + nk_B T) v) = 0. \quad (4.14)$$

4.2. First Order Expansion of the Distribution Function

At 0 order in λ , the system is at equilibrium and all f_i are Maxwellians with the same velocity and temperature. We can thus write

$$f_i = \bar{M}_i + O(\lambda) \quad (4.15)$$

where \bar{M}_i is the maxwellian with moments $n^{(i)}$, v , T (remember from the definition (2.6), (2.8), (2.8) that v and T are the averaged quantities of the mixture).

Now, rewriting (3.1)

$$f_i = M_i - \frac{1}{v_i} (\partial_t f_i + \xi \cdot \nabla_x f_i) \quad (4.16)$$

and using (4.15) we can write

$$f_i = M_i - \frac{1}{v_i} (\partial_t \bar{M}_i + \xi \cdot \nabla_x \bar{M}_i) + O(\lambda^2) \tag{4.17}$$

4.3. First Order Expansion of Partial Momenta

Integrating (4.17) over $m_i \xi d\xi$ gives

$$m_i v_i n^{(i)} v^{(i)} = m_i v_i n^{(i)} v_i - m_i \int_{\mathbb{R}^3} (\partial_t \bar{M}_i + \xi \cdot \nabla_x \bar{M}_i) \xi d\xi + O(\lambda^2) \tag{4.18}$$

To go further we need a preliminary computation

Lemma 4.4.

$$m_i \int_{\mathbb{R}^3} (\partial_t \bar{M}_i + \xi \cdot \nabla_x \bar{M}_i) \xi d\xi = \nabla_x (n^{(i)} k_B T) - \frac{\rho^{(i)}}{\rho} \nabla_x (n k_B T)$$

Proof of Lemma 4.4. This quantity I can be computed explicitly,

$$I = \partial_t (\rho^{(i)} v) + \text{div} (n^{(i)} k_B T \underline{Id} + \rho^{(i)} v \otimes v)$$

$$\rho I = \rho \nabla_x (n^{(i)} k_B T) + \rho v \partial_t \rho^{(i)} + \rho \rho^{(i)} \partial_t v + \rho (v \otimes v) \nabla_x \rho^{(i)} + \rho \rho^{(i)} \text{div} (v \otimes v)$$

Subtracting $\rho^{(i)}$ times Eq. (4.13) we obtain

$$\begin{aligned} \rho I &= \rho \nabla_x (n^{(i)} k_B T) - \rho^{(i)} \nabla_x (n k_B T) + \rho v \partial_t \rho^{(i)} - \rho^{(i)} v \partial_t \rho + \rho (v \otimes v) \nabla_x \rho^{(i)} \\ &\quad - \rho^{(i)} (v \otimes v) \nabla_x \rho \end{aligned}$$

Also, rewriting (4.13), it is easy to check

$$v \partial_t \rho^{(i)} + (v \otimes v) \nabla_x \rho^{(i)} = -v \rho^{(i)} \text{div} v$$

and the same formula is true for ρ (because the total mass is conserved as well). Then

$$\rho I = \rho \nabla_x (n^{(i)} k_B T) - \rho^{(i)} \nabla_x (n k_B T) - \rho v \rho^{(i)} \text{div} v + \rho^{(i)} v \rho \text{div} v$$

and the result follows.

Now we rewrite (4.18) and use the lemma, this rises

$$-m_i v_i n^{(i)} v^{(i)} + m_i v_i n^{(i)} v_i = I = \nabla_x (n^{(i)} k_B T) - \frac{\rho^{(i)}}{\rho} \nabla_x (n k_B T) + O(\lambda^2) \tag{4.19}$$

The right hand side can be written

$$I = \rho^{(i)} \frac{\nabla_x(n^{(i)}k_B T)}{\rho^{(i)}} - \frac{\rho^{(i)}}{\rho} \sum_{k=1}^n \rho^{(k)} \frac{\nabla_x(n^{(k)}k_B T)}{\rho^{(k)}} \quad (4.20)$$

Therefore, denoting by Ω the symmetric matrix

$$\Omega_{ik} = \rho^{(i)} \delta_{ik} - \frac{\rho^{(i)} \rho^{(k)}}{\rho}$$

and, for simplicity

$$\alpha_k = \frac{\nabla_x(n^{(k)}k_B T)}{\rho^{(k)}}$$

we have

$$I = \sum_{k=1}^n \Omega_{ik} \alpha_k \quad (4.21)$$

The left hand side of (4.19) can be written, using the definition of v_i in (3.3):

$$\begin{aligned} I &= n^{(i)} \sum_{k=1}^n 2\mu_{ik} \chi_{ik} n^{(k)} (v^{(k)} - v^{(i)}) \\ I &= \sum_{k=1}^n \frac{2\chi_{ik}}{m_i + m_k} \rho^{(i)} \rho^{(k)} \left(\frac{J_k}{\rho^{(k)}} - \frac{J_i}{\rho^{(i)}} \right) \end{aligned}$$

So that we can write

$$I = \sum_{k=1}^n M_{ik} J_k \quad (4.22)$$

where M is the matrix defined by:

$$M_{ik} = \frac{2\chi_{ik} \rho^{(i)}}{m_i + m_k} - \left(\sum_{k'=1}^n \frac{2\chi_{ik'} \rho^{(k')}}{m_i + m_{k'}} \right) \delta_{ik}$$

Now we conclude from the two expressions (4.22) and (4.21) that

$$M \cdot J = \Omega \cdot \alpha \quad (4.23)$$

At this point it is useful to notice that the matrix M is not necessarily invertible. In fact, denoting for simplicity by A_{ik} the symmetric coefficient

$$A_{ik} = \frac{2\chi_{ik}}{m_i + m_k}$$

the matrix M is the following

$$\begin{aligned} M_{ik} &= - \sum_{k' \neq i} A_{ik'} \rho^{(k')} & \text{if } i = k \\ M_{ik} &= A_{ik} \rho^{(i)} & \text{if } i \neq k \end{aligned}$$

We want to use the following classical property:

The matrices T with the following property

$$|T_{ii}| > \sum_{k \neq i} |T_{ki}| \quad \text{for every } i \quad (4.24)$$

are always invertible.

Here we are in the case of

$$\begin{aligned} |M_{ii}| &= \sum_{k \neq i} |M_{ki}| & \text{for every } i \\ M_{ii} &< 0, \quad M_{ik} > 0 & \text{if } i \neq k \end{aligned}$$

Thus, taking a small parameter $0 < \kappa < \text{minimum}_{i \neq k} M_{ik}$, the matrix \tilde{M} defined by

$$\tilde{M}_{ik} = M_{ik} - \kappa$$

verifies the property (4.24) and is invertible.

Now we remark that

$$\sum_i J_i = 0$$

which is a straightforward consequence of the definition of J_i and v and expresses the conservation of total momentum.

It is easy to check from this that adding a constant to M everywhere does not change the result,

$$\tilde{M} \cdot J = M \cdot J = \Omega \cdot \alpha$$

and we conclude by the invertibility of \tilde{M} that

$$J = (\tilde{M}^{-1}\Omega) \cdot \alpha$$

This prove the relation (4.7) with $L = -\tilde{M}^{-1}\Omega$.

To prove the Onsager relation, that is the symmetry of $L = -\tilde{M}^{-1}\Omega$ we first prove that $M \cdot \Omega$ is symmetric.

$$(M \cdot \Omega)_{ij} = \sum_{k=1}^n \left(\left(A_{ik} \rho^{(i)} - \sum_{k'=1}^n A_{ik'} \rho^{(k')} \delta_{ik} \right) \left(\rho^{(k)} \delta_{kj} - \frac{\rho^{(j)} \rho^{(k)}}{\rho} \right) \right)$$

$$(M \cdot \Omega)_{ij} = A_{ij} \rho^{(i)} \rho^{(j)} - \left(\sum_{k'=1}^n A_{ik'} \rho^{(k')} \right) \rho^{(j)} \delta_{ij} - \sum_k A_{ik} \frac{\rho^{(i)} \rho^{(j)} \rho^{(k)}}{\rho} \\ + \sum_{k'} A_{ik'} \frac{\rho^{(k')} \rho^{(j)} \rho^{(i)}}{\rho}$$

$$(M \cdot \Omega)_{ij} = A_{ij} \rho^{(i)} \rho^{(j)} - \left(\sum_{k'=1}^n A_{ik'} \rho^{(k')} \right) \rho^{(j)} \delta_{ij}$$

which is symmetric.

Next we remark that $\kappa \cdot \Omega = 0$ (where κ denotes here the matrix with all terms equal to κ). This is a simple computation:

$$(\kappa \cdot \Omega)_{ij} = \sum_k \kappa \left(\rho^{(j)} \delta_{kj} - \frac{\rho^{(j)} \rho^{(k)}}{\rho} \right) = \kappa \rho^{(j)} - \kappa \rho^{(j)} \frac{\sum_k \rho^{(k)}}{\rho} = 0$$

We thus conclude that $\tilde{M} \cdot \Omega = M \cdot \Omega$ is symmetrical. Recalling that Ω is also symmetrical, and denoting by \tilde{M}^t the transposed matrix of \tilde{M} ,

$$\Omega \cdot \tilde{M}^t = \tilde{M} \cdot \Omega$$

gives

$$\tilde{M}^{-1} \cdot \Omega = \Omega \cdot (\tilde{M}^t)^{-1} = (\tilde{M}^{-1} \cdot \Omega)^t$$

And we conclude that $\tilde{M}^{-1} \cdot \Omega$ is symmetrical, and thus L is symmetrical.

4.4. First Order Expansion of the Total Stress and Heat Flux

The expansion of P is obtained in a similar way. From (4.17) one obtains

$$P = \bar{P} - \sum_i \frac{m_i}{v_i} \int_{\mathbb{R}^3} (\partial_t \bar{M}_i + \xi \cdot \nabla_x \bar{M}_i) (\xi - v) \otimes (\xi - v) d\xi + O(\lambda^2)$$

where

$$\bar{P} = \sum_i \int_{\mathbb{R}^3} m_i M_i(\xi - v) \otimes (\xi - v) d\xi$$

A computation (see, for example, ref. 3 for a similar computation) gives

$$\begin{aligned} m_i \int_{\mathbb{R}^3} (\partial_i \bar{M}_i + \xi \cdot \nabla_x \bar{M}_i)(\xi - v) \otimes (\xi - v) d\xi \\ = n^{(i)} k_B T (\nabla_x u + (\nabla_x u)^t - \frac{2}{3} (\operatorname{div} u) \underline{Id}) \end{aligned}$$

and

$$P = \bar{P} - \eta (\nabla_x u + (\nabla_x u)^t - \frac{2}{3} (\operatorname{div} u) \underline{Id}) + O(\lambda^2) \tag{4.25}$$

where η is the viscosity coefficient

$$\eta = k_B T \sum_i \frac{n^{(i)}}{v_i}$$

Now we compute \bar{P}

$$\bar{P} = \sum_i \left(\frac{2}{3} n^{(i)} \varepsilon_i \underline{Id} + \rho^{(i)} (v_i - v) \otimes (v_i - v) \right)$$

Since at equilibrium $v_i = v$, we have $v_i - v = O(\lambda)$ and thus we deduce that \bar{P} is diagonal at first order,

$$\bar{P} = \left(\sum_i \frac{2}{3} n^{(i)} \varepsilon_i \right) \underline{Id} + O(\lambda^2)$$

Now we can take the trace in (4.25), and remark that the viscosity term has a nul trace. Thus we have

$$\sum_i n^{(i)} \varepsilon_i = \sum_i n^{(i)} \varepsilon^{(i)} + O(\lambda^2)$$

From the definition of ε , we obtain

$$n\varepsilon = \sum_i \left(n^{(i)} \varepsilon^{(i)} + \frac{\rho^{(i)}}{2} |v^{(i)} - v|^2 \right) = \sum_i n^{(i)} \varepsilon^{(i)} + O(\lambda^2)$$

and we conclude that

$$\begin{aligned}\bar{P} &= \frac{2}{3} n \varepsilon \underline{Id} + O(\lambda^2) \\ P &= n k_B T \underline{Id} - \eta (\nabla_x u + (\nabla_x u)^t - \frac{2}{3} (\operatorname{div} u) \underline{Id}) + O(\lambda^2)\end{aligned}\quad (4.26)$$

For the heat flux, one obtains in a similar way

$$q = \bar{q} - \kappa \nabla_x T + O(\lambda^2) \quad (4.27)$$

where κ is the heat flux

$$\kappa = \frac{5}{2} k_B^2 T \sum_i \frac{n^{(i)}}{m_i v_i}$$

and

$$\begin{aligned}\bar{q} &= \sum_i \int_{\mathbb{R}^3} m_i M_i(\xi - v) \frac{|\xi - v|^2}{2} d\xi \\ \bar{q} &= \sum_i \int_{\mathbb{R}^3} m_i M_i(\xi - v_i + v_i - v) \frac{|\xi - v_i + v_i - v|^2}{2} d\xi\end{aligned}$$

Since M_i is an even function of $(\xi - v_i)$, in the expansion the terms of first and third order in $(\xi - v_i)$ vanish, and the term of zero order is $\rho^{(i)} |v_i - v|^3 = O(\lambda^3)$. We thus obtain:

$$\bar{q} = \sum_i \left(1 + \frac{2}{3}\right) n^{(i)} \varepsilon_i (v_i - v) + O(\lambda^2)$$

To conclude, we use $\varepsilon_i = \varepsilon + O(\lambda)$ and $n^{(i)}(v_i - v) = \frac{J_i}{m_i} = O(\lambda)$, this gives

$$\bar{q} = \sum_i \frac{5}{3} \varepsilon \frac{J_i}{m_i} + O(\lambda^2)$$

and finally

$$q = \frac{5}{2} k_B T \sum_i \frac{J_i}{m_i} - \kappa \nabla_x T + O(\lambda^2) \quad (4.28)$$

5. APPENDIX 1

We compute here the exchange coefficients which arise in the various “conservation laws” that can be deduced from the Boltzmann equation

$$\frac{\partial}{\partial t} f_i + \xi \cdot \nabla_x f_i = Q_i(f, f)$$

As usual, these are obtained in multiplying the Boltzmann equation by 1, ξ and $|\xi|^2$ and integrating $d\xi$. We first obtain, thanks to (2.5), the conservation of mass

$$\frac{\partial}{\partial t} n^{(i)} + \text{div}(n^{(i)}v^{(i)}) = 0 \tag{5.1}$$

The second balance law concerns momentum and contains exchange terms. For maxwellian molecules, i.e.,

$$B_{ik}(n.V, |V|) = \bar{B}_{ik}(n.V/|V|)$$

with \bar{B}_{ik} an even function, we find

$$\frac{\partial}{\partial t} (\rho^{(i)}v^{(i)}) + \text{div} \Sigma^{(i)} = \sum_{k=1}^n 2\mu_{ik}\chi_{ik}n^{(i)}n^{(k)}[v^{(k)} - v^{(i)}] \tag{5.2}$$

where $\chi_{ik} = \chi_{ki}$ is a proportionality factor between forces

$$\chi_{ik} = \int_{B_+} \left(\frac{n.V}{|V|}\right)^2 \bar{B}_{ik}\left(\frac{n.V}{|V|}\right) dn \tag{5.3}$$

To see this, we just compute the righthand side which is given, thanks to the relations (2.5), by

$$\begin{aligned} & m_i \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{B_+} \xi(f'_i f'_{k^*} - f_i f_{k^*}) \bar{B}_{ik}(n.V/|V|) d\xi d\xi_* dn \\ &= m_i \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{B_+} f_i f_{k^*} (\xi' - \xi) \bar{B}_{ik}(n.V/|V|) d\xi d\xi_* dn \\ &= 2\mu_{ik} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{B_+} f_i f_{k^*} n [(\xi_* - \xi) \cdot n] \bar{B}_{ik}(n.V/|V|) d\xi d\xi_* dn \\ &= 2\mu_{ik} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{B_+} f_i f_{k^*} (\xi_* - \xi)(n.V/|V|)^2 \bar{B}_{ik}(n.V/|V|) d\xi d\xi_* dn \end{aligned}$$

and the result follows.

As for the energy balance, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} E^{(i)} + \operatorname{div} q^{(i)} &= \sum_{k=1}^n 2\mu_{ik}\chi_{ik}n^{(i)}n^{(k)} \left[(v^{(k)} - v^{(i)}) \cdot v^{(i)} \right. \\ &\quad \left. + \frac{2}{m_i + m_k} \left(\varepsilon^{(k)} - \varepsilon^{(i)} + m_k \frac{|v^{(k)} - v^{(i)}|^2}{2} \right) \right] \end{aligned} \quad (5.4)$$

The derivation is again based on the use of relations (2.5). We have

$$\begin{aligned} &m_i \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{B_+} \frac{|\xi|^2}{2} (f'_i f'_{k^*} - f_i f_{k^*}) \bar{B}_{ik}(n \cdot V/|V|) d\xi d\xi_* dn \\ &= m_i \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{B_+} f_i f_{k^*} \left(\frac{|\xi'|^2}{2} - \frac{|\xi|^2}{2} \right) \bar{B}_{ik}(n \cdot V/|V|) d\xi d\xi_* dn \\ &= 2m_i \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{B_+} f_i f_{k^*} \left[\left(\frac{\mu_{ik}}{m_i} \right)^2 |(\xi_* - \xi) \cdot n|^2 + \frac{\mu_{ik}}{m_i} \xi \cdot n [(\xi_* - \xi) \cdot n] \right] \\ &\quad \times \bar{B}_{ik}(n \cdot V/|V|) d\xi d\xi_* dn \\ &= 2m_i \chi_{ik} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_i f_{k^*} \left[\left(\frac{\mu_{ik}}{m_i} \right)^2 |\xi_* - \xi|^2 + \frac{\mu_{ik}}{m_i} \xi \cdot (\xi_* - \xi) \right] d\xi d\xi_* \\ &= 2m_i \chi_{ik} n^{(i)} n^{(k)} \left[\left(\frac{\mu_{ik}}{m_i} \right)^2 \left(|v^{(k)} - v^{(i)}|^2 + 2 \frac{\varepsilon^{(k)}}{m_k} + 2 \frac{\varepsilon^{(i)}}{m_i} \right) \right. \\ &\quad \left. + \frac{\mu_{ik}}{m_i} \left[(v^{(k)} - v^{(i)}) \cdot v^{(i)} - 2 \frac{\varepsilon^{(i)}}{m_i} \right] \right] \end{aligned}$$

and the result follows from algebraic relations between m_i and μ_{ik} .

6. APPENDIX 2

This appendix is dedicated to the proof of the equilibrium result of Section 2.5, following the ideas of ref. 13.

We first recall the notations (2.2)

$$\begin{cases} \xi' = \xi - \frac{2\mu_{ij}}{m_i} n [(\xi - \xi_*) \cdot n] \\ \xi_*' = \xi_* + \frac{2\mu_{ij}}{m_j} n [(\xi - \xi_*) \cdot n] \end{cases} \quad (6.1)$$

We also recall that the micro-collision operator

$$T_n: (\xi, \xi_*) \mapsto (\xi', \xi'_*)$$

is an involution and

$$d\xi d\xi_* = d\xi' d\xi'_* \tag{6.2}$$

Proposition 6.1. We suppose that f_i and f_j are distributions such that

$$f_i f_{j^*} = f'_i f'_{j^*}, \quad f_j f_{i^*} = f'_j f'_{i^*}$$

for all n in B_+ . Then f_i and f_j are Maxwellians with same velocity and temperature.

Proof of Proposition 6.1. We denote by g_i, g_j the Fourier transforms of f_i, f_j .

With the above hypothesis we can write

$$g_i(k) g_j(k_*) = \int f_i(\xi') f_j(\xi'_*) e^{i(k \cdot \xi' + k_* \cdot \xi'_*)} d\xi' d\xi'_*$$

Now we change the notation ξ' by ξ , use the involution property and $d\xi' d\xi'_* = d\xi d\xi_*$ to obtain

$$\begin{aligned} g_i(k) g_j(k_*) &= \int f_i(\xi) f_j(\xi_*) e^{i(k \cdot \xi + k_* \cdot \xi_*)} d\xi d\xi_* \\ &= \int f_i(\xi) f_j(\xi_*) e^{i(k \cdot \xi + k_* \cdot \xi_*) - i\left(\frac{2\mu_{ij}}{m_i} k - \frac{2\mu_{ij}}{m_j} k_*\right) \cdot n(\xi - \xi_*) \cdot n} d\xi d\xi_* \end{aligned}$$

Now we can consider this as a formula in n (this is true for all n) and make a Taylor expansion around n_0 where n_0 is orthogonal to $\frac{2\mu_{ij}}{m_i} k - \frac{2\mu_{ij}}{m_j} k_*$. We write $n = n_0 + \eta$ and the first term of the Taylor expansion is

$$0 = \int f_i(\xi) f_j(\xi_*) e^{i(k \cdot \xi + k_* \cdot \xi_*)} i \left(\frac{2\mu_{ij}}{m_i} k - \frac{2\mu_{ij}}{m_j} k_* \right) \cdot \eta(\xi - \xi_*) \cdot n_0 d\xi d\xi_*$$

This gives

$$\int f_i(\xi) f_j(\xi_*) e^{i(k \cdot \xi + k_* \cdot \xi_*)} (\xi - \xi_*) \cdot n_0 d\xi d\xi_* = 0, \quad \forall n_0 \perp \frac{k}{m_i} - \frac{k_*}{m_j}$$

wich also reads

$$n_0 \cdot (\nabla_k g_i g_{j^*} - \nabla_{k_*} g_i g_{j^*}) = 0, \quad \forall n_0 \perp \frac{k}{m_i} - \frac{k_*}{m_j} \quad (6.3)$$

Now, by galilean invariance, we may assume that the mean velocity of f_i vanishes, that is

$$\int f_i(\xi) \xi d\xi = 0 = \nabla g_i(0)$$

Then taking $k = 0$ in (6.3) gives for all k_*

$$g_i(0) n_0 \cdot \nabla_{k_*} g_{j^*} = 0, \quad \forall n_0 \perp k_*$$

This means that $\nabla_{k_*} g_{j^*}$ is proportionnal to k_* , and thus that g_j is a radial function

$$g_{j^*} = g_j(|k_*|^2)$$

Now, if $k_* = |k_*| n$ where n is a unit vector, $\nabla_{k_*} g_{j^*}$ is parallel to n , and taking $k_* = 0$, $\nabla_{k_*} g_{j^*}(0)$ is parralel to any unit vector thus vanishes,

$$\nabla_{k_*} g_{j^*}(0) = 0$$

Remark that this gives the first result that f_j has the same zero bulk velocity as f_i .

We can now take $k_* = 0$ in (6.3) to obtain $g_i = g_i(|k|^2)$.

From now on we denote $|k|^2 = r$ and $|k_*|^2 = r_*$, and compute from (6.3):

$$k \frac{g'_i(r)}{g_i(r)} - k_* \frac{g'_j(r_*)}{g_j(r_*)}$$

is orthogonal to n_0 and thus proportionnal to

$$\frac{1}{m_i} k - \frac{1}{m_j} k_*$$

Thus one obtains that

$$m_j \frac{g'_j(r_*)}{g_j(r_*)} = m_i \frac{g'_i(r)}{g_i(r)} \quad (6.4)$$

We conclude that $\frac{g_i'}{g_i} = (\ln(g_i))'$ is independent of r , so that

$$g_j(k) = C_j e^{-\alpha_j |k|^2}$$

and in the same way

$$g_i(k) = C_i e^{-\alpha_i |k|^2}$$

Moreover we also deduce from (6.3) that

$$\alpha_j = \frac{m_i}{m_j} \alpha_i$$

Then since the Fourier transform of a gaussian law is a gaussian law, we obtain that f_i and f_j are Maxwellians with zero velocity and same temperature.

ACKNOWLEDGMENTS

The authors wish to thank Professor Y. Sone for pointing out the indifferentiability principle and suggesting to improve the known models, and for his hospitality during the visit in Kyoto of two of the authors. They also wish to thank B. Dubroca for fruitful discussions.

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